

Arithmetic Properties of Curves Related to Quadratic Iteration

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Abstract

We use Chebyshev polynomials to construct a family of hyperelliptic curves defined over the rational numbers that have complex multiplication and have superspecial reduction modulo a set of primes with positive density, and we characterize their rational points. This construction produces infinitely many curves of arbitrarily large genus, whose Jacobians are of positive rank and geometrically simple, and whose rational points have been completely determined. As an application, we compute the decomposition (into simple factors) and rational torsion subgroups of a more general class of Jacobians that arise in the study of quadratic iteration and dynamical Galois theory. Building upon previous work in [10], we show that the Hall-Lang conjecture on integral points of elliptic curves implies a finite index result for the Galois groups of iterates of $x^2 + c$, and we use conjectural bounds for the Mordell curves to predict the index in the still unknown case when $c = 3$. Finally, we use Chabauty's method in combination with the Mordell-Weil sieve to establish results about the Galois behavior of the the fourth iterate of such quadratic polynomials.

1 Introduction

The arithmetic properties of iterated rational maps: heights, orbits, Galois theory etc. provide many interesting problems, both directly and by analogy, in arithmetic geometry. Even in the most studied and basic case, the quadratic polynomial, much remains a mystery.

Let us begin by fixing some notation. Suppose that $f = f_c = x^2 + c$ for $c \in \mathbb{Q}$, and denote by f^n the n^{th} iterate of f , that is, the n -fold composition of f with itself. One can ask about the arithmetic properties of $\{f(x_0), f^2(x_0), f^3(x_0) \dots\}$, the orbit of x_0 . For example, we may ask how c influences which prime factors divide elements of the orbit. We focus on the simpler question: for which c can the orbit of a point contain a square? Viewed another way, we wish

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to understand the rational points on $\mathfrak{C}_n : y^2 = f_c^n(x)$. Some examples where squares do occur in an orbit include $(c, x, y, n) = (-78, 25/2, 311/4, 2)$ and $(-21, 5, 2, 5)$.

This curve, when we view c as fixed, is very much related to the factorization of f^n , or more generally its Galois theory. For instance, it is known that f^n is irreducible if the set $\{-c, f^2(0), f^3(0) \dots\}$ does not contain a rational square; see [13]. In the case when c is an integer, f^n is irreducible for all n provided only that $-c$ is not a rational square. Hence our curves are nonsingular (more specifically one can choose a nonsingular curve with \mathfrak{C}_n as an affine model).

In Section 2 we discuss various arithmetic properties of \mathfrak{C}_n and a related curve $B_n : y^2 = (x - c) \cdot f_c^n(x)$, showing that for $c = -2$ these curves have complex multiplication and superspecial reduction. With these results in place, we address the factorization of the Jacobians of \mathfrak{C}_n for more general c and determine their rational torsion subgroups. Furthermore, we completely characterize the rational points on B_n when $c = -2$. This construction produces infinitely many curves of arbitrarily large genus, whose Jacobians are of positive rank and geometrically simple, and whose rational points have been completely determined. We also discuss possible generalizations of this method to other post critically finite polynomials.

Linking the arithmetic properties of these curves to the Galois theory of f^n , we prove a “large” image theorem in Section 3 using standard conjectures on the size of integral points on elliptic curves. This result is an analogy of a theorem of Serre for non CM elliptic curves [2].

Remark 0.1. Assuming that the ABC conjecture holds over \mathbb{Q} , this result was proven independently by the author in [10] and by Gratton, Nguyen, and Tucker in [9].

Finally, we predict which rational values of c supply a polynomial whose fourth iterate is the first to have smaller than expected Galois group. The only examples up to a very large height correspond to $c = 2/3$ and $c = -6/7$. Moreover, we prove that there are no such integer values (in contrast to the $n = 3$ case in [10]) and formulate questions for larger n .

All Galois groups were computed with Sage [18], and the descent calculations were carried out with Magma [1].

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2 Reduction to Chebyshev Polynomials and Complex Multiplication

When studying the rational points on a curve of large genus, one often attempts to find a map to a curve of lower genus, since the typical arithmetic procedures are more easily carried out in this setting. Note that by iterating f we obtain many maps $\mathfrak{C}_n \rightarrow \mathfrak{C}_{n-1} \rightarrow \dots \rightarrow \mathfrak{C}_1$. However, in order to completely decompose the Jacobians of \mathfrak{C}_n , we wish to find maps to simple curves. Fortunately, for $m < n$ we have the coverings

$$B_m : y^2 = (x - c) \cdot f_c^m(x), \quad \pi_m : \mathfrak{C}_n \rightarrow B_m, \quad \pi_m(x, y) = (f_c^{n-m}(x), y \cdot f_c^{n-m-1}(x)).$$

It will be shown that for many values of c and every m , the Jacobian $J(B_m)$ of B_m is simple. We will adopt the notation $J(C)$ for the Jacobian of any curve C throughout.

To do this, we will extract information from the very special case when $c = -2$. In this situation, $f = x^2 - 2$ is a Chebyshev polynomial of degree 2, often denoted T_2 elsewhere in the literature [19]. More generally, f^n is the Chebyshev polynomial T_{2^n} of degree 2^n .

We consider the Chebyshev polynomials T_d as characterized by the equation

$$T_d(z + z^{-1}) = z^d + z^{-d}, \quad \text{for all } z \in \mathbb{C}^*.$$

T_d is known to be a degree d monic polynomial with integer coefficients. The classical Chebyshev polynomials \tilde{T}_d were defined in the following way:

$$\text{If we write } z = e^{it}, \text{ then } \tilde{T}_d(2 \cos(t)) = 2 \cos(dt),$$

though we will use the first characterization where T_d is monic. For a more complete discussion of these polynomials, see [19].

Our heuristic will be the following: prove the special local and global properties for the curves B_n in the Chebyshev case, and then extract information about the more general situation by reducing to this case. For example, if $c = 63$, then $f \equiv T_2 \pmod{5}$ and $f \equiv T_2 \pmod{13}$. It follows from Theorem 1 that $J(B_n)$ is simple for all n and $J(B_n)(\mathbb{Q})_{\text{Tor}} \cong \mathbb{Z}/2\mathbb{Z}$ for $n \leq 30$. With this in mind, we have the following definition.

Definition 0.1. We say that f_c has *Chebyshev reduction* modulo p if $f_c \equiv T_2 \pmod{p}$.

Unless otherwise stated, we will assume $c = -2$ for the remainder of this section. The main result over finite fields is the following:

Theorem 1. *For $n \geq 1$ set $B_n^\pm := y^2 = (x \pm 2) \cdot f^n(x)$. If $p \equiv 5 \pmod{8}$, then B_n^\pm is superspecial over \mathbb{F}_p , meaning that the characteristic polynomial of Frobenius is $t^{2^n} + p^{2^{n-1}}$. In particular, $J(B_n)$ is supersingular and $J(B_n)(\mathbb{F}_p) \cong \mathbb{Z}/(p^{2^{n-1}} + 1)\mathbb{Z}$ is cyclic for all n .*

Proof. First note that B_n is nonsingular when viewed over \mathbb{F}_p for every odd prime. This follows from the fact that $f = T_2$ is critically finite: $\{f(0), f^2(0), f^3(0) \dots\} = \{\pm 2\}$. Hence the discriminant of $(x \pm 2) \cdot f^n(x)$ is a power of 2. In general, the discriminant Δ_m of f_c^m for any quadratic polynomial $f_c = x^2 + c$ is:

$$\Delta_m = \pm \Delta_{m-1}^2 \cdot 2^{2^m} \cdot f_c^m(0);$$

see Lemma 2.6 of [13]. In order to proceed, we are in need of the following elementary lemma.

Lemma 1.1. *If $m < 2^n$, then the field \mathbb{F}_{p^m} contains an element α satisfying $\alpha^{2^{n+1}} = 1$, which is not a square in \mathbb{F}_{p^m} .*

Proof. Write $q = p^m$ and suppose that $m = 2^t$. Notice that \mathbb{F}_q contains an element α of order 2^{t+2} , since $p^{2^t} - 1$ is divisible by 2^{t+2} and \mathbb{F}_q^* is cyclic. To see this, write

$$(p^{2^t} - 1) = (p^{2^{t-1}} - 1) \cdot (p^{2^{t-1}} + 1) = (p - 1) \cdot (p + 1) \cdot (p^2 + 1) \cdots (p^{2^{t-1}} + 1),$$

inductively. As $p - 1 \equiv 0 \pmod{4}$ and every other term in the product is even, we see that $p^{2^t} - 1$ is divisible by 2^{t+2} .

However, $p \equiv 5 \pmod{8}$ implies that no higher power of 2 can divide the product. Hence α is not a square in \mathbb{F}_q . Finally, the conditions on m force $t \leq n - 1$. Therefore, $\alpha^{2^{n+1}} = 1$ as desired.

In general we may write $m = 2^t \cdot a$ for some odd a . By applying the result in the 2-powered case to the subfield $\mathbb{F}_{p^{2^t}} \subset \mathbb{F}_q$, we may find an element $\alpha \in \mathbb{F}_{p^{2^t}}$ with the desired properties.

Note that if such an element α were a square in \mathbb{F}_q , then it must be a square in $\mathbb{F}_{p^{2^t}}$ (otherwise there would be a proper quadratic extension of $\mathbb{F}_{p^{2^t}}$ contained in \mathbb{F}_q , contradicting the fact that a is odd). However, as was the case above, the fact that $p \equiv 5 \pmod{8}$ implies that α is not a square in $\mathbb{F}_p^{2^t}$. The result follows in general. \square

Now for the proof of the Theorem. Consider the auxiliary curve

$$C_n := y^2 = x \cdot (x^{2^n} + 1),$$

which is equipped with the maps

$$\phi_{\pm} : C_{n+1} \rightarrow B_n^{\pm}, \quad \phi_{\pm}(x, y) = \left(x + \frac{1}{x}, \frac{(x \pm 1) \cdot y}{x^{2^{n+1}+1}} \right).$$

One checks that ϕ_{\pm} is a quotient map for the involution

$$\psi_{\pm}(x, y) = \left(\frac{1}{x}, \frac{\pm y}{x^{2^n+1}} \right),$$

of C_{n+1} respectively. If $G = \text{Aut}(C_{n+1})$, then a result of Kani and Rosen on the equivalence of idempotents in the group algebra $\mathbb{Q}[G]$ implies that the Jacobian of the curve C_{n+1} has a decomposition:

$$J(C_{n+1}) \sim J(B_n^+) \times J(B_n^-),$$

over \mathbb{Q} ; see [16]. Here we write $A \sim B$ if A and B are isogenous abelian varieties.

Since every odd prime is a prime of good reduction, we have a similar decomposition of $J(C_n)$ over \mathbb{F}_p . Therefore, it will suffice to compute the Euler polynomial of C_{n+1} , respectively the characteristic of Frobenius, to find that of B_n .

We will denote the Euler polynomial and the characteristic polynomial of Frobenius by L_p and χ respectively. For the formulas relating the coefficients of these polynomials to the number of points on the curve over \mathbb{F}_q , see §14.1 in [7].

Note that the genus of C_{n+1} is 2^n , and so to compute the coefficients of the Euler polynomial of C_{n+1} , we need to find

$$N_m = \#C_{n+1}(\mathbb{F}_q) = q + 1 + \sum_{x \in \mathbb{F}_q} \left(\frac{x}{q} \right) \left(\frac{x^{2^{n+1}} + 1}{q} \right),$$

for $q = p^m$ and $m \leq 2^n$. To do this, suppose that $m < 2^n$ and apply the lemma to find an $\alpha \in \mathbb{F}_{p^m}$ which is not a square and satisfies $\alpha^{2^{n+1}} = 1$. Then

$$N_m - (p^m + 1) = \sum_{x \in T} \sum_{i=1}^{2^{n+1}} \left(\frac{x \cdot \alpha^i}{q} \right) \left(\frac{(\alpha^i \cdot x)^{2^{n+1}} + 1}{q} \right) = \sum_{x \in T} \left(\sum_{i=1}^{2^{n+1}} \left(\frac{\alpha^i}{q} \right) \right) \left(\frac{x}{q} \right) \left(\frac{x^{2^{n+1}} + 1}{q} \right) = 0,$$

where T is a set of coset representatives for $\mathbb{F}_q^*/\langle \alpha \rangle$. It follows that $\chi(t) = t^{2^{n+1}} + at^{2^n} + p^{2^n}$ for some integer a .

We will show that $a = 2p^{2^{n-1}}$, since then $\chi(t) = (t^{2^n} + p^{2^{n-1}})^2$ and $t^{2^n} + p^{2^{n-1}}$ is irreducible over \mathbb{Q} . Unique factorization of polynomials over a field would then imply that B_n^{\pm} is superspecial. To see that $t^{2^n} + p^{2^{n-1}}$ is irreducible, make a change of variables $t \rightarrow z + 1$ and use Eisenstein's criteria at the prime 2.

However, B_n^+ and B_n^- are quadratic twists of one another by -1 . Since $p \equiv 5 \pmod{8}$ (in particular $p \equiv 1 \pmod{4}$), we have that $J(C_{n+1}) \sim J(B_n^+)^2$ and $L_p(C_n, t) = L_p(B_n^+, t)^2$. As $\chi(t) = t^{2^{n+1}} + at^{2^n} + p^{2^n}$, the Weil bound implies that $|a| \leq 2 \cdot p^{2^{n-1}}$. Moreover, if we write

$$a = 2p^{2^{n-1}} + 2b_1^2 \cdot p^{2^{n-1}-1} + \dots b_{2^{n-2}}^2,$$

where the b_i 's are the coefficients of $L_p(B_n^+, t)$, then the bound on a implies that

$$2b_1^2 \cdot p^{2^{n-1}-1} + \dots b_{2^{n-2}}^2 = 0.$$

We conclude that $b_i = 0$ for all i as desired.

The group structure of $J(B_n)(\mathbb{F}_p)$ follows from a theorem of Zhu, which may be found in §45 of [11] \square

Having characterized $J(B_n)$ over \mathbb{F}_p for $p \equiv 5 \pmod{8}$, we can similarly describe \mathfrak{C}_n and $J(\mathfrak{C}_n)$ over \mathbb{F}_p for such p . We summarize our knowledge in the following corollary.

Corollary 1.1. *Let $\chi_{p,n}$ denote the characteristic polynomial of Frobenius when we view \mathfrak{C}_n over \mathbb{F}_p . If $p \equiv 5 \pmod{8}$, then for all $n \geq 2$:*

$$\chi_{p,n}(t) = t^{2^n-2} + pt^{2^n-4} + p^2t^{2^n-6} + \dots + p^{2^{n-1}-2}t^2 + p^{2^{n-1}-1}.$$

In particular, $J(\mathfrak{C}_n)$ is supersingular and

$$\#J(\mathfrak{C}_n)(\mathbb{F}_p) = p^{2^{n-1}-1} + p^{2^{n-1}-2} + \dots p + 1.$$

Proof. Use the characterization of $L_p(B_i, t)$ in the theorem and the fact that $L_p(\mathfrak{C}_n, t) = \prod_{i=1}^{n-1} L_p(B_i, t)$, since B_n is simple over \mathbb{F}_p . The result follows. \square

Using our theorem in the Chebyshev case, we now gather some global information about $J(\mathfrak{C}_n)_{/\mathbb{Q}}$ for many integer values of c . Further still, we characterize completely the rational torsion subgroup $J(B_n)(\mathbb{Q})_{\text{Tor}}$ when $f = T_2$, summarized below.

Corollary 1.2. *Suppose that $f(x) = x^2 + c$ and define $\mathfrak{C}_n := y^2 = f^n(x)$. If $c \equiv -2 \pmod{p}$ for some $p \equiv 5 \pmod{8}$ (for instance one can take $c = -2, 3, 11, 27, \dots, p-2$), then*

$$J(\mathfrak{C}_n) \sim J(B_1) \times J(B_2) \times \dots J(B_{n-1})$$

is a decomposition into \mathbb{Q} -simple factors for every $n \geq 1$. Furthermore, if $c = -2$, then $J(B_n)(\mathbb{Q})_{\text{Tor}} \cong \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 1$ and $J(B_n)(\mathbb{Q})$ has positive rank for all $n \geq 2$.

Proof. For the first statement, one reduces modulo p and uses Theorem 1 to find that the Euler factor at p is irreducible over \mathbb{Q} (Eisenstein at 2). It follows that the $J(B_m)$ are simple over \mathbb{F}_p for every m , and hence simple over \mathbb{Q} as well.

For the second statement, we again assume $c = -2$ and f^n is a Chebyshev polynomial. Note that $J(B_n)(\mathbb{Q})$ certainly contains an element of order two coming from the Weierstrass point $(2, 0)$ on B_n . On the other hand, the prime to p part of $J(B_n)(\mathbb{Q})_{\text{Tor}}$ injects into $J(B_n)(\mathbb{F}_p)$ via the reduction map. Hence, $J(B_n)(\mathbb{Q})_{\text{Tor}} \cong \mathbb{Z}/2\mathbb{Z}$ follows from the fact that the

$$\gcd(5^{2^n} + 1, 13^{2^n} + 1, 29^{2^n} + 1, \dots) = 2 \text{ for all } n,$$

where the above set ranges over almost all primes $p \equiv 5 \pmod{8}$.

To see this, fix n and suppose that p^* is an odd prime which divides $p^{2^n} + 1$ for almost all $p \equiv 5 \pmod{8}$. In particular $p^{2^n} \equiv -1 \pmod{p^*}$, and hence $p^* \equiv 1 \pmod{4}$.

Now note that the $\gcd(4p^* + 1, 8p^*) = 1$, so that Dirichlet's theorem on arithmetic progressions implies that there exist infinitely many primes p_0 with $p_0 = 4p^* + 1 + 8p^*k_0$ for some integer k_0 . Moreover, since $4p^* + 1 \equiv 5 \pmod{8}$, we have that $p_0 \equiv 5 \pmod{8}$. Hence, we may choose p_0 such that $p_0^{2^n} + 1 \equiv 0 \pmod{p^*}$ by our assumption on p^* .

Finally, one sees that $1 \equiv 4p^* + 1 + 8p^*k_0 \equiv p_0 \pmod{p^*}$ and $2 \equiv p_0^{2^n} + 1 \equiv 0 \pmod{p^*}$. This is a contradiction since p^* is odd.

On the other hand $f^n(0) = 2$ for all $n \geq 2$, from which it follows that $(0, 2) \in B_n(\mathbb{Q})$. Moreover, it cannot be a torsion point by the argument above. This finishes the proof. \square

Remark 1.1. Evidence suggests that generically our decomposition of $J(\mathfrak{C}_n)$ is into geometrically simple factors. To check this, we use a result of Leprevost: if there exists a prime of good reduction p such that the Galois group of the characteristic polynomial of Frobenius at p is the full $S_g \times (\mathbb{Z}/2\mathbb{Z})^g$, then the Jacobian of the curve is absolutely simple; see [15].

Of course in practice we can use Chebyshev reduction to show that $J(B_n)(\mathbb{Q})_{\text{Tor}} \cong \mathbb{Z}/2\mathbb{Z}$ for many values of c and many n . For instance, one easily verifies that $\gcd(5^{2^n} + 1, 13^{2^n} + 1) = 2$ for all $n \leq 30$. Hence, even without the aid of almost all $p \equiv 5 \pmod{8}$, if $c = 63 + k65$ for some integer k , then $J(B_n)(\mathbb{Q})_{\text{Tor}} \cong \mathbb{Z}/2\mathbb{Z}$ for such n . The greatest common divisors above were computed with Sage.

In the $c = -2$ case, the positive density of primes at which $J(B_n)$ has supersingular (super-special) reduction suggests the presence of latent symmetries. Indeed, one checks that $J(B_1)$ is an elliptic curve having complex multiplication by $\mathbb{Z}[\sqrt{-2}]$:

$$[\sqrt{-2}](x, y) = \left(-1/2 \cdot \frac{x^2 - 2}{x - 2} + 2, \frac{1}{-2\sqrt{-2}} \cdot \frac{y \cdot ((x - 2)^2 - 2)}{(x - 2)^2} \right).$$

Moreover, by checking Igusa invariants against known examples, one also sees that $J(B_2)$ complex multiplication by $\mathbb{Q}(\sqrt{\sqrt{2} - 2})$. The Theorem below, which follows from the work of Carocca, Lange, and Rodriguez [5], shows that these examples are no accident and provide a general construction of hyperelliptic curves, defined over the rational numbers, having complex multiplication.

Theorem 2. *If ζ is a primitive 2^{n+2} -th root of unity and $d = 2^{n+1} - 1$, then $J(B_n)$ is an absolutely simple abelian variety that has complex multiplication by $\mathbb{Q}(\zeta + \zeta^d)$.*

Proof. Notice that C_{n+1} has complex multiplication by $\mathbb{Q}(\zeta)$, induced by the map

$$[\zeta] : C \rightarrow C, \quad [\zeta](x, y) = (\zeta^2 x, \zeta y).$$

We have already seen that the quotient curve of C_{n+1} by the automorphism ψ is B_n , and $J(C_n) \sim J(B_n)^2$ over $\bar{\mathbb{Q}}$. It follows that the simple factors of $J(B_n)$ have complex multiplication by some subfield of $\mathbb{Q}(\zeta)$; see Chapter 1 of [14].

In [5, Theorem 2] it was shown that the curve $C'_{n+1} : y^2 = x(x^{2^{n+1}} - 1)$ has a quotient X with the property that $J(X)$ has complex multiplication by $\mathbb{Q}(\zeta + \zeta^d)$. Moreover, since $\mathbb{Q}(\zeta + \zeta^d)$ does not contain any proper CM fields, $J(X)$ must be absolutely simple; see [14].

Remark 2.1. This was done by studying the general case of metacyclic Galois coverings $Y \rightarrow \mathbb{P}^1$ branched at 3-points, building upon previous work of Ellenberg. To translate, the relevant Galois covering group is

$$G = \langle [\zeta], \psi \mid [\zeta]^{2^{n+1}} = \psi^2 = 1, \psi \circ [\zeta] \circ \psi = [\zeta^d] \rangle,$$

and one can take Y to be C'_{n+1} .

However, note that C_{n+1} and C'_{n+1} are twists, becoming isomorphic over $\mathbb{Q}(\zeta)$. Since any decomposition of an abelian variety is unique up to isogeny, we must have that $J(X) \sim J(B_n)$. Hence, $\text{End}_0(J(X)) \cong \text{End}_0(J(B_n))$ and $J(B_n)$ has CM as claimed. \square

We conclude this investigation of the curves defined by Chebyshev polynomials by characterizing their rational points, summarized in the following Theorem.

Theorem 3. *We have the following characterization in the case when $c = -2$:*

$$B_n(\mathbb{Q}) = \{\infty, (-2, 0), (0, \pm 2)\}, \text{ for all } n \geq 2,$$

$$\mathfrak{C}_n(\mathbb{Q}) = \{\infty^\pm\}, \text{ for all } n \geq 2,$$

$$C_n(\mathbb{Q}) = \{\infty, (0, 0)\}, \text{ for all } n \geq 1.$$

Proof. If $n \geq 2$, then \mathfrak{C}_n maps to $B_1 : y^2 = (x+2)(x^2-2)$. However, B_1 is an elliptic curve, and a 2-descent shows that $B_1(\mathbb{Q})$ has rank zero. It follows that $B_1(\mathbb{Q}) = \{\infty, (-2, 0)\}$, and after computing preimages, we see that $\mathfrak{C}_n(\mathbb{Q})$ contains only the infinite points.

If $n = 2$, a 2-descent shows that the rank of $J(B_2)(\mathbb{Q})$ is one. Moreover, after running the Chabauty function in Magma, we see that $B_2(\mathbb{Q}) = \{\infty, (-2, 0), (0, \pm 2)\}$. This matches our claim for larger n . For the remaining $n \geq 3$, we use covering collections to determine $B_n(\mathbb{Q})$.

Since the resultant of $x+2$ and f^n is equal to 2 (for all n), the rational points on B_n are covered by the rational points on the curves

$$D_n^{(d)} : du^2 = x+2, \quad dv^2 = f^n(x), \quad \text{for } d \in \{\pm 1, \pm 2\};$$

see Example 9 of [22]. We will proceed by examining the second defining equation $\mathfrak{C}_n^{(d)} : dv^2 = f^n(x)$ of $D_n^{(d)}$. If $d = 1$, then our description of $\mathfrak{C}_n(\mathbb{Q})$ implies that $D_n(\mathbb{Q})$ has only the points at infinity. If $d = -2$, then $\mathfrak{C}_n^{(-2)}$ maps to the elliptic curve $B_1^{(-2)} : -2v^2 = (x+2) \cdot (x^2-2)$ via $(x, y) \rightarrow (f^{n-1}(x), -2 \cdot f^{n-2}(x) \cdot y)$. A descent shows that $B_1^{(-2)}(\mathbb{Q})$ has rank zero, from which it easily follows that $B_1^{(-2)}(\mathbb{Q}) = \{\infty, (-2, 0)\}$. By computing preimages, we find that $\mathfrak{C}_n^{(-2)}$, and hence $D_n^{(-2)}$, has no rational points.

For the remaining cases when $d = -1$ and $d = 2$, we map $\mathfrak{C}_n^{(d)}$ to $B_2^{(d)}$ via $(x, y) \rightarrow (f^{n-1}(x), d \cdot f^{n-2}(x) \cdot y)$. However, in either scenario, we find that $J(B_2^{(d)})(\mathbb{Q})$ has rank one, and moreover, we can compute a generator using bounds between the Weil and canonical heights. After running the Chabauty function in Magma and computing preimages, we find that $\mathfrak{C}_n^{(2)}(\mathbb{Q}) = \{(0, \pm 1), (\pm 2, \pm 1)\}$ and $\mathfrak{C}_n^{(-1)}(\mathbb{Q}) = \{(\pm 1, \pm 1)\}$ for $n \geq 3$. It follows that $D_n^{(2)}(\mathbb{Q}) = \{(0, \pm 1, \pm 1), (-2, 0, \pm 1)\}$ and $D_n^{(-1)}(\mathbb{Q})$ is empty. Moreover, we see that $B_n(\mathbb{Q}) = \{\infty, (-2, 0), (0, \pm 2)\}$ as claimed.

Finally, If $n = 1$, then C_n is an elliptic curve of rank zero having only 2-torsion. One easily verifies its rational points are Weierstrass points. To determine $C_n(\mathbb{Q})$ for $n \geq 2$, map C_n to B_{n-1} via ϕ_+ as in Theorem 1. We have already determined $B_{n-1}(\mathbb{Q})$, and so by computing preimages, we conclude that $C_n(\mathbb{Q}) = \{\infty, (0, 0)\}$. \square

Possible Generalization: Notice that we have determined the rational points on infinitely curves B_n that do not cover any lower genus curves (their Jacobians have complex multiplication by a CM field with no proper CM subfields). However, because the orbit of -2 is finite under the PCF polynomial $x^2 - 2$, the rational points on B_n are covered by finitely many computable twists of a curve D_n , each of which do map to many lower genus curves. In general, if $g \in \mathbb{Z}[x]$ is a post critically finite polynomial and a is an integer such that the orbit $\{a, g(a), g^2(a) \dots\}$ is finite, then one should in principle be able to find the rational points on $C_{n,g}(a) : y^2 = (x - a) \cdot g^n(x)$, using similar methods as in the theorem. Of course, we must also assume that the iterates of g are separable, to be assured that this equation is nonsingular.

Having studied some of the properties of curves defined by quadratic iteration, we transition to the related problem of determining the Galois groups of the polynomial f^n . In fact, the relationship between $\text{Gal } f^n$ and \mathfrak{C}_n motivated our investigation of these curves in the first place. In keep with our earlier conventions, we will continue to suppress c in the notation.

3 Dynamical Galois Groups and Curves

In order to probe how the curves \mathfrak{C}_n and their quadratic twists relate to the Galois theory of f^n , we must first discuss the necessary background. Generically, suppose that $f \in \mathbb{Q}[x]$ is a polynomial of degree d whose iterates are separable (the polynomials obtained from successive composition of f have distinct roots in an algebraic closure). If T_n denotes the set of roots of f, f^2, \dots, f^n together with 0, then T_n carries a natural d -ary rooted tree structure: $\alpha, \beta \in T_n$ share an edge if and only if $f(\alpha) = \beta$. As f is a polynomial with rational coefficients, the Galois group of f^n , which we denote by $\text{Gal}(f^n)$, acts via graph automorphisms on T_n . Such a framework provides an arboreal representation, $\text{Gal}(f^n) \hookrightarrow \text{Aut}(T_n)$, and we can ask about the size of the image (note that $\text{Aut}(T_n)$ is the n -fold iterated wreath product of $\mathbb{Z}/d\mathbb{Z}$).

In the quadratic case, it has been conjectured that the image of $\text{Gal}(f^n)$ is “large”, under mild assumptions on f : if all iterates of f are irreducible and f is critically infinite, then it has been conjectured that the image of the inverse limit, $G(f) := \varprojlim \text{Gal}(f^n)$, is of finite index in the automorphism group of the full preimage tree $\text{Aut}(T)$, (here critically infinite means the orbit of the unique root of f ’s derivative, also known as critical point, is infinite). This is an analog of Serre’s result for the Galois action on the prime-powered torsion points of a non CM elliptic curve. For a nice expository on this problem, see [2].

These hypothesis are easily satisfied in the case when $f(x) = x^2 + c$ and c is an integer: If $c \neq -2$ and $-c$ is not a square, then f^n is irreducible for all n , and the set $\{f(0), f^2(0), \dots\}$ is infinite. Stoll has given congruence relations on c which ensure that the Galois groups of iterates of $f(x) = x^2 + c$ are as large as possible [21]. However, much is unknown as to the behavior of integer values not meeting these criteria, not to mention the more general setting of rational c (for instance $c = 3$ and $-2/3$).

In order to attack the finite index conjecture for more general values of c , we use the following fundamental lemma, which relates the index of the Galois groups of successive iterates to the rational points on certain curves; see [21, Corollary 1.3] for the case when $f = x^2 + c$ and

c is an integer or [13, Lemma 3.2] for the general result and proof:

Lemma 3.1. *Let $f \in \mathbb{Q}[x]$ be a quadratic polynomial, γ its unique critical point (root of f 's derivative), and K_m be a splitting field for f^m . If f, f^2, \dots, f^n are all irreducible polynomials, then the subextension K_n/K_{n-1} is not maximal if and only if $f^n(\gamma)$ is a square in K_{n-1} .*

As promised, we use quadratic twists of \mathfrak{C}_n and the Hall-Lang conjecture on integral points of elliptic curves to prove the finite index conjecture in this case when c is an integer.

Theorem 4. *Suppose that $f(x) = x^2 + c$ for some integer c not equal to -2 . If $-c$ is not a square, then the Hall-Lang conjecture on integral points of elliptic curves implies that $|\text{Aut}(T) : \text{Gal}(f)|$ is finite. Moreover, if the weak form of Hall's conjecture for the Mordell curves holds with $C = 50$ and exponent 4 (i.e $\epsilon = 2$ and $C(\epsilon) = 50$), then when $f(x) = x^2 + 3$ we have that $|\text{Aut}(T) : \text{Gal}(f)| = 2$.*

Proof. Let $f(x) = x^2 + c$. Suppose that the subextension K_n/K_{n-1} is not maximal, or more precisely that $|K_n/K_{n-1}| \neq 2^{2^{n-1}}$. It is our goal to show that n is bounded.

By Lemma 2.1, this is equivalent to $f^n(0) \in (K_{n-1})^2$, and hence for some $y \in \mathbb{Z}$

$$dy^2 = f^n(0), \text{ with } \mathbb{Q}(\sqrt{d}) \subset K_{n-1};$$

Moreover, $d = \prod_i p_i$, where the p_i 's are distinct primes dividing $2 \cdot \prod_{j=1}^{n-1} f^j(0)$. To see this latter fact, we use the formula for the discriminant Δ_m of f^m ,

$$\Delta_m = \pm \Delta_{m-1}^2 \cdot 2^{2^m} \cdot f^m(0),$$

given in Lemma 2.6 of [13]. It follows that the only rational primes which ramify in K_{n-1} are the primes dividing $2 \cdot \prod_{j=1}^{n-1} f^j(0)$. Since the primes which divide d must ramify in K_{n-1} , we obtain the desired description of the p_i . Also note that if $p_i | f^j(0)$ and $p_i | f^n(0)$ (which is the case since $p_i | d$), then it divides $f^{n-j}(0)$.

In any case we can have the refinement that $d = \prod_i p_i$, where the p_i 's are distinct primes dividing $2 \cdot \prod_{j=1}^{\lfloor n/2 \rfloor} f^j(0)$, where $\lfloor x \rfloor$ denotes the floor function.

A rational point on the curve $\mathfrak{C}_n^{(d)} := dy^2 = f^n(x)$ maps to $B_1^{(d)} := dy^2 = (x - c) \cdot f(x)$ via $(x, y) \rightarrow (f^{n-1}(x), y \cdot f^{n-2}(x))$. Transforming $B_1^{(d)}$ into standard form, we get

$$E_1^{(d)} : y^2 = x^3 + 598752(c^2 - 3c)d^2x + 161243136(c^3 - 18c^2)d^3,$$

via $(x, y) \rightarrow (d \cdot (x - 12c), 2d^2y)$. In particular, if K_n/K_{n-1} is not maximal, then we obtain an integer point

$$(d \cdot (f^{n-1}(0) - 12c), 2yd^2 \cdot f^{n-2}(0)) \in E_1^{(d)}(\mathbb{Q}).$$

If we assume the Hall-Lang conjecture for integral points on elliptic curves, then there exist constants C and K such that

$$d \cdot (f^{n-1}(0) - 12c) < C \cdot \max\{|598752(c^2 - 3c)d^2|, |161243136(c^3 - 18c^2)d^3|\}^K.$$

See [20] or [6] for the relevant background in elliptic curves. In either case

$$|f^{n-1}(0)| < C' \cdot |d|^{K'} \leq C' |f(0) \cdot f^2(0) \cdot \dots \cdot f^{\lfloor n/2 \rfloor}(0)|^{K'},$$

for some new constants C' and K' . However, this implies that n is bounded.

For example, if $c > 0$, then $f^m(0) > f(0) \cdot f^2(0) \dots f^{m-1}(0)$ for all m . Hence, if we let $t = \lfloor n/2 \rfloor + 1$ and suppose $K' < 2^s$, then

$$f^{n-1}(0) < C' \cdot (f^t(0))^{K'} < C'(f^t(0))^{K'} < C' f^{t+s}(0),$$

from which the result follows. A similar argument works in the case $c \leq -3$, we simply use the fact that $|f^m(0)| \geq (f^{m-1}(0) - 1)^2$, as seen in [21, Corollary 1.3].

Explicitly, when $c = 3$ the j invariant of $B_1^{(d)}$ is zero, and we may transform $B_1^{(d)}$ into the Mordell curve $M^{(-2d)^3} := y^2 = x^3 - (2d)^3$. In particular, a point $(0, y) \in \mathfrak{C}_n^{(d)}(\mathbb{Q})$ yields a point

$$((f^{n-1}(0) - 1) \cdot d, d^2 \cdot y \cdot f^{n-2}(0)) \in M^{(-2d)^3}(\mathbb{Q}).$$

If the weak form of Hall's conjecture for the Mordell Curves holds with $\epsilon = 2$ and $C(\epsilon) = 50$, then for $f(x) = x^2 + 3$ we have that

$$|(f^{n-1}(0) - 1) \cdot d| < 50 \cdot |(-2d)^3|^4.$$

This implies that

$$f^{n-1}(0) < 204800d^{11} + 1 \leq 204800(f^{\lfloor n/2 \rfloor + 1}(0))^{11} + 1 < 204800(f^{\lfloor n/2 \rfloor + 5}(0)) + 1.$$

However, such a bound implies that $n \leq 13$. Moreover, one checks that the only $n \leq 13$ with $dy^2 = f^n(0)$ and $d = \prod_i p_i$, where the p_i 's are distinct primes dividing $2 \cdot \prod_{j=1}^{n-1} f^j(0)$, is $n = 3$. In this case, $f^3(0) = 7^2 \cdot 3 = 7^2 \cdot f(0)$ and $|\text{Aut}(T_3) : \text{Gal}(f^3)| = 2$. It follows that the index of the entire family $|\text{Aut}(T) : \text{Gal}(f)|$ must also equal 2. \square

Remark 4.1. For the exponent 4, our constant $C = 50$ safely fits the data pitting the known integer points on the Mordell curves against the size of their defining coefficients; see [8]. In fact, even for Elkies' large examples, our choice works (thus far) for the strong form of Hall's conjecture (exponent of 2).

A different approach: To this point we have used the curves coming from $f(x) = x^2 + c$ to study the Galois theory of f 's iterates (viewing $c \in \mathbb{Q}$ as fixed), hoping to say something about the stability as n grows. We now change our perspective slightly. Suppose we fix a stage n , and ask for which rational values of c is the Galois group of f^n smaller than expected (Hilbert's irreducibility theorem assures us that such a set of rational numbers is thin). Of course this question needs to be refined, as there will be many trivial values of c (for instance if $-c$ is a square). A natural adjustment then is to ask for which rational numbers is $\text{Gal}(f^n)$ smaller than expected for the first time at stage n . As noted in Theorem 3, an interesting example is $c = 3$ and $n = 3$:

$$\text{Gal}((x^2 + 3)^2 + 3) \cong D_4 \cong \text{Aut}(T_2),$$

yet one computes that $|\text{Gal}(((x^2 + 3)^2 + 3)^2 + 3)| = 64 < 2^{2^3 - 1}$, and hence the third iterate of $f = x^2 + 3$ is the first of f 's iterates to have a Galois group which is not maximal (not equal to the full automorphism group of the preimage tree). This leads to the following definition:

Definition 4.1. Suppose $n \geq 2$ and that $c \in \mathbb{Q}$. If $f_c(x) = x^2 + c$ is a quadratic polynomial such that $\text{Gal}(f^{n-1}) \cong \text{Aut}(T_{n-1})$ yet $\text{Gal}(f^n) \not\cong \text{Aut}(T_n)$, then we say that f has a *small* n -th iterate. Furthermore let

$$S^{(n)} := \{c \in \mathbb{Q} \mid f_c \text{ has a small } n\text{-th iterate}\},$$

be the set of rational values of c supplying a polynomial with small n -th iterate.

Our refined question then becomes to describe $S^{(n)}$. In the case when $n = 3$, the author completely characterizes $S^{(3)}$ in terms of the x -coordinates of two rank-one elliptic curves; see [10]. In particular, using bounds on elliptic logarithms, we concluded that $S^{(3)} \cap \mathbb{Z} = \{3\}$, and hence the example above is the only such integer example for $n = 3$.

We can use this characterization to compute many new examples of polynomials with small third iterate by adding together the points corresponding to known examples on the elliptic curve: e.g. $f(x) = x^2 - 2/3, x^2 + 6/19, x^2 - 17/14$. Moreover, since generators of both curves are easily computed, we have in some sense found all examples.

It is natural to ask what happens for larger n . As an illustration, consider the case when $n = 4$ and $c = -6/7$. One computes that

$$|\text{Gal}(((x^2 - 6/7)^2 - 6/7)^2 - 6/7)| = 2^{23-1} \quad \text{and} \quad |\text{Gal}((((x^2 - 6/7)^2 - 6/7)^2 - 6/7)^2 - 6/7)| = 8192.$$

Since $8192 < 2^{24-1}$, we see that $-6/7 \in S^{(4)}$. Are there any other examples? Although not entirely satisfactory, we have the following theorem:

Theorem 5. *There are no integer values of c in $S^{(4)}$. Furthermore, if the curve $y^2 = x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1$ does not have any rational points of height greater than 10^{100} , then $S^{(4)} = \{2/3, -6/7\}$.*

Proof. As in [10], we associate values in $S^{(n)}$ with the rational points on certain curves. To continue, we are in need of the following lemma:

Lemma 5.1. *Let $f_c(x) = f(x) = x^2 + c$ and K_m be a splitting field for f^m . If $c \in S^{(4)}$, then f^4 is irreducible and*

$$\mathbb{Q}(\sqrt{-c}), \mathbb{Q}(\sqrt{f^2(0)}), \mathbb{Q}\left(\sqrt{-\frac{f^2(0)}{c}}\right), \mathbb{Q}(\sqrt{f^3(0)}), \mathbb{Q}\left(\sqrt{-\frac{f^3(0)}{c}}\right), \mathbb{Q}\left(\sqrt{\frac{f^3(0)}{f^2(0)}}\right), \mathbb{Q}\left(\sqrt{-\frac{f^3(0)}{c+1}}\right)$$

are the distinct quadratic subfields of K_3 .

Proof. First note that if $\text{Gal}(f^m) \cong \text{Aut}(T_m)$, then K_m contains exactly $2^m - 1$ quadratic subfields. The reason is that the number of quadratic subfields is the number of subgroups of $\text{Gal}(K_m)$ whose quotient is $\mathbb{Z}/2\mathbb{Z}$. Now $\text{Aut}(T_m)$ is the m -fold wreath product of $\mathbb{Z}/2\mathbb{Z}$, and one can show that the maximal abelian quotient of exponent 2 of this group is $\mathbb{Z}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z}$, where the product is m -fold; see [21]. This quotient group, by its maximality property, will contain as a subgroup any quotient that is abelian of exponent 2, and hence the quotients that are isomorphic to $\mathbb{Z}/2\mathbb{Z}$ are in one-to-one correspondence with the subgroups of $(\mathbb{Z}/2\mathbb{Z})^m$ of order 2. However, that is the same as the number of distinct elements of order 2, which is $2^m - 1$.

Now for the proof of the lemma. Suppose that $c \in S^{(4)}$. Then $\text{Gal}(f^j) \cong \text{Aut}(T_j)$ for all $1 \leq j \leq 3$ and the subextensions K_3/K_2 , K_2/K_1 and K_1/\mathbb{Q} are all maximal. In particular

$-f(0) = -c$ is not a rational square and $K_1 = \mathbb{Q}(\sqrt{-c})$. Since $\text{Aut}(T_2)$ acts transitively on the roots of f^2 , it follows that f^2 is irreducible. Then Lemma 3.2 implies that $f^2(0) \notin (K_1)^2$. In particular $\mathbb{Q}(\sqrt{-c})$, $\mathbb{Q}(\sqrt{f^2(0)})$ and $\mathbb{Q}\left(\sqrt{\frac{f^2(0)}{-c}}\right)$ are distinct quadratic subfields of K_2 (if $m \geq 2$, then discriminant formula above implies that $\sqrt{f^m(0)} \in K_m$). By the opening remarks in the proof of the Lemma, there must be exactly $2^2 - 1$ or 3 such subfields. Hence our list is exhaustive for K_2 .

One simply repeats this argument for the third iterate, obtaining the claimed list of quadratic subfields of K_3 . In particular the set $\{-f(0), f^2(0), f^3(0)\}$ does not contain a rational square. It suffices to show that $f^4(0)$ is also not a rational square to deduce that f^4 is irreducible; see [13]. However, a 2-descent on the curve

$$F_0 : y^2 = f_c^4(0) = ((c^2 + c)^2 + c)^2 + c$$

shows that its Jacobian has rank zero. After reducing modulo several primes of good reduction, one finds that any torsion must be 2-torsion. Hence $F_0(\mathbb{Q}) = \{\infty^\pm, (0, 0), (-1, 0)\}$, and $c \in S^{(4)}$ implies that $c \neq 0, -1$. \square

With the lemma in place, we are ready to relate elements of $S^{(4)}$ with the rational points on certain curves. If $c \in S^{(4)}$ then f_c^4 is irreducible and $f_c^4(0)$ is not a rational square (see the proof of the Lemma 4.1). However, since the extension K_4/K_3 is not maximal by assumption, Lemma 3.1 implies that $f_c^4(0) \in (K_3)^2$. In particular $\sqrt{f_c^4(0)} \in K_3$, so that $\sqrt{f_c^4(0)}$ must live in one of the seven quadratic subfields of K_3 listed above. Moreover, there must exist y such that (c, y) is a rational point on one of the curves:

$$F_1 : y^2 = \frac{f_x^4(0)}{-x} = -(x^7 + 4x^6 + 6x^5 + 6x^4 + 5x^3 + 2x^2 + x + 1),$$

$$F_2 : y^2 = \frac{f_x^4(0)}{f_x^2(0)} = x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1,$$

$$F_3 : y^2 = \frac{f_x^4(0)}{-(x+1)} = -x(x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1),$$

$$F_4 : y^2 = \frac{f_x^4(0)}{x} \cdot \frac{f_x^3(0)}{x} = (x^7 + 4x^6 + 6x^5 + 6x^4 + 5x^3 + 2x^2 + x + 1) \cdot (x^3 + 2x^2 + x + 1),$$

$$F_5 : y^2 = f_x^4(0) \cdot \frac{f_x^3(0)}{-x} = (x^8 + 4x^7 + 6x^6 + 6x^5 + 5x^4 + 2x^3 + x^2 + x) \cdot -(x^3 + 2x^2 + x + 1),$$

$$F_6 : y^2 = \frac{f_x^4(0)}{f_x^2(0)} \cdot f_x^3(0) = (x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1) \cdot (x^4 + 2x^3 + x^2 + x),$$

$$F_7 : y^2 = \frac{f_x^4(0)}{-(f_x^2(0))} \cdot \frac{f_x^3(0)}{x} = -(x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1) \cdot (x^3 + 2x^2 + x + 1),$$

(after dividing out the finite singularities coming from $c = 0, -1$ when necessary).

Note that all of these curves are hyperelliptic, and so at least in principal, their arithmetic: ranks, integer points, etc. are more easily computable. Also note that the interesting rational points corresponding to known elements of $S^{(4)}$ both come from F_2 . They are $(2/3, 53/27)$ and $(-6/7, 377/343)$ respectively.

Therefore, to describe $S^{(4)}$ it will suffice to characterize $F_i(\mathbb{Q})$ for all $1 \leq i \leq 7$. We will do this sequentially, employing standard methods in the theory of rational points on curves. For a nice overview of these techniques, see [22].

1: A 2-descent with Magma shows that the rational points on the Jacobian of F_1 have rank one [1]. Moreover, reducing modulo various primes of good reduction, one sees that the order of any rational torsion point must divide 4. However, the only 2-torsion point is $T = [(-1, 0) - \infty]$ and by examining the image of T via the 2-descent map, one sees that T is not a double in $J(F_1)(\mathbb{Q})$. It follows that $J(F_1)(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. Since F_1 has genus three, we can apply the method of Chabauty and Coleman to bound the rational points (in fact, find them all).

To do this we change variable to obtain an equation which is more amenable to the computations to come: send $(x, y) \rightarrow (-x - 2, y)$ to map to the curve

$$F'_1 : y^2 = x^7 + 10x^6 + 42x^5 + 94x^4 + 117x^3 + 76x^2 + 21x + 1.$$

A naive point search yields $\{\infty, (-1, 0), (0, \pm 1)\} \subseteq F'_1$ and we will show this set is exhaustive.

Let $J = J(F'_1)$ and use the point $P_0 = (0, 1)$ to define an embedding of $F'_1(\mathbb{Q}) \subseteq J(\mathbb{Q}) \subseteq J(\mathbb{Q}_3)$ via $Q \rightarrow [Q - P]$. Then given a 1-form ω on $J(\mathbb{Q}_3)$, one can integrate to form the function

$$J(\mathbb{Q}_3) \rightarrow \mathbb{Q}_3, \quad \text{given by} \quad P \rightarrow \int_0^P \omega.$$

Coleman's idea was to notice that if we restrict this function to a residue class of $F'_1(\mathbb{Q}_3) \subseteq J(\mathbb{Q}_3)$, then this function can be computed explicitly in terms of power series (using a parametrization coming from a uniformizer for the class). Moreover, in the case when the genus is larger than the rank of the group $J(\mathbb{Q})$, as is the case in our example, one can find an ω where the above function vanishes on $J(\mathbb{Q})$. Finally, using Newton polygons, one can bound the number of rational points in each residue class by bounding the number of zeros of a power series in \mathbb{Z}_3 . This is what we will do. For a nice expository on this method, see [17].

We will follow the notation and outline of Example 1 in Section 9 of Wetherell's thesis [23]. In particular, we use x as a local coordinate system on the residue class at $(0, 1)$ and the basis $\eta_0 = (1/y)dx$, $\eta_1 = (x/y)dx$, and $\eta_2 = (x^2/y)dx$ for the global forms on F'_1 . Expanding $1/y$ in a power series in terms of x we get:

$$\eta_0 = \frac{dx}{y} = 1 - \frac{21}{2}x + \frac{1019}{8}x^2 - \frac{28089}{16}x^3 + \frac{3292019}{128}x^4 - \frac{99637707}{256}x^5 + \frac{6153979535}{1024}x^6 \dots$$

Furthermore, it is known that the η_i are in $\mathbb{Z}_3[[x]]$. Then we have the integrals λ_i for the η_i in the residue class of $(0, 1)$; that is

$$\lambda_i(P) = \int_{(0,1)}^P \eta_i.$$

From our formulas for the η_i , we have:

$$\begin{aligned} \lambda_0 &= x - \frac{21}{4}x^2 + \frac{1019}{24}x^3 - \frac{28089}{64}x^4 + \frac{3292019}{640}x^5 - \frac{33212569}{512}x^6 + \frac{6153979535}{7168}x^7 - \dots \\ \lambda_1 &= \frac{1}{2}x^2 - \frac{7}{2}x^3 + \frac{1019}{32}x^4 - \frac{28089}{80}x^5 + \frac{3292019}{768}x^6 - \frac{99637707}{1792}x^7 + \frac{6153979535}{8192}x^8 - \dots \end{aligned}$$

$$\lambda_2 = \frac{1}{3}x^3 - \frac{21}{8}x^4 + \frac{1019}{40}x^5 - \frac{9363}{32}x^6 + \frac{3292019}{896}x^7 - \frac{99637707}{2048}x^8 + \frac{6153979535}{9216}x^9 + \dots$$

Let ω_i be the differentials on J corresponding to the η_i on F'_1 , i.e. the pullbacks relative to the inclusion $F'_1(\mathbb{Q}_3) \subseteq J(\mathbb{Q}_3)$ given by $P \rightarrow [P - (0, 1)]$. Finally, let λ'_i be the homomorphism from $J(\mathbb{Q}_3)$ to \mathbb{Q}_3 obtained by integrating the ω_i . We will calculate the λ'_i on $J_1(\mathbb{Q}_3)$, the kernel of the reduction map.

Let $a \in J_1(\mathbb{Q}_3)$, so that a may be represented as $a = [P_1 + P_2 + P_3 - 3P_0]$ with $P_i \in C(\bar{\mathbb{Q}}_3)$ and $\bar{P}_i = \bar{P}_0 = (0, 1)$. If $s_j = \sum_{i=1}^3 x(P_i)^j$, then from the expression

$$\int_0^a \omega_i = \sum_j \int_{(0,1)}^{P_j} \eta_i$$

we see that

$$\begin{aligned} \lambda_0 &= s_1 - \frac{21}{4}s_2 + \frac{1019}{24}s_3 - \frac{28089}{64}s_4 + \frac{3292019}{640}s_5 - \frac{33212569}{512}s_6 + \frac{6153979535}{7168}s_7 - \dots \\ \lambda_1 &= \frac{1}{2}s_2 - \frac{7}{2}s_3 + \frac{1019}{32}s_4 - \frac{28089}{80}s_5 + \frac{3292019}{768}s_6 - \frac{99637707}{1792}s_7 + \frac{6153979535}{8192}s_8 - \dots \\ \lambda_2 &= \frac{1}{3}s_3 - \frac{21}{8}s_4 + \frac{1019}{40}s_5 - \frac{9363}{32}s_6 + \frac{3292019}{896}s_7 - \frac{99637707}{2048}s_8 + \frac{6153979535}{9216}s_9 + \dots \end{aligned}$$

We wish to find an ω such that its integral kills $J(\mathbb{Q})$. However, since $\log(J(\mathbb{Q}))$ has rank one in $T_0(J(\mathbb{Q}_3))$, the dimension of such differentials is 2, and so we have some freedom with our choice. We will exploit this freedom to bound the number of points in each residue field.

Note that if $U = [\infty - P_0]$, then $12U$ is in $J_1(\mathbb{Q}_3)$. Furthermore, a 1-form kills $J(\mathbb{Q})$ if and only if it kills $12U$, since the index of the subgroup generated by the rational torsion and U is coprime to $|J(\mathbb{F}_3)| = 24$. Using Magma, we calculate the divisor $12U$ represented as $[P_1 + P_2 + P_3 - 3P_0]$ where the 3 symmetric functions in the $x(P_i)$ are

$$\begin{aligned} \sigma_1 &= \frac{5688167583876464940561144764011383197382945288}{5528939601706074645413409528185601232466043121} \equiv 2 \cdot 3^4 \pmod{3^5}, \\ \sigma_2 &= \frac{-2183647192786560140353830791558556354713308560}{5528939601706074645413409528185601232466043121} \equiv 2^2 \cdot 3^3 \pmod{3^5}, \\ \sigma_3 &= \frac{4352156372570507181684433225178910249832181376}{5528939601706074645413409528185601232466043121} \equiv 2^3 \cdot 3^3 \pmod{3^5}. \end{aligned}$$

Choosing a precision of 3^5 was arbitrary, though sufficient for our purposes. Note that the valuation of every $x(P_i)$ is at least $\min\{v(\sigma_i)/i\} = \min\{4, 3/2, 1\} = 1$. It follows that $v(s_j) \geq j$. Moreover, one verifies that every term past $j = 3$ of $\lambda_i(12U)$ is congruent to 0 mod 3^5 .

After calculating the s_j in terms of the σ_i , one has that

$$\begin{aligned} \lambda_0(12U) &\equiv 2 \cdot 3^3 \pmod{3^5}, \\ \lambda_1(12U) &\equiv 3^3 \pmod{3^5}, \\ \lambda_2(12U) &\equiv 2^3 \cdot 3^3 \pmod{3^5}. \end{aligned}$$

As the integral is linear in the integrand, there exist global one forms α and β such that

$$\int_0^P \alpha = 0 \quad \text{and} \quad \int_0^P \beta = 0 \quad \text{for all } P \in J(\mathbb{Q}),$$

with $\alpha \equiv 2\lambda_1 - \lambda_0 \pmod{3^5}$ and $\beta \equiv 2\lambda_2 - 4\lambda_0 \pmod{3^5}$. Moreover, we have that $\alpha = (x-1)dx/y$ and $\beta = (x^2-1)dx/y$ when we view them over \mathbb{F}_3 . However, for every $P \in C(\mathbb{F}_3)$, either α or β does not vanish at P ; see the table below.

$P \in C(\mathbb{F}_p)$	$\text{ord}_P(\alpha)$	$\text{ord}_P(\beta)$
∞	2	0
$(0, 1)$	0	0
$(0, -1)$	0	0
$(-1, 0)$	0	1

It follows that every residue class in $C(\mathbb{Q}_3)$ contains at most one rational point, and hence exactly one rational point as claimed.

2: We first use Runge's method to find $F_2(\mathbb{Z})$. This involves completing the square. Suppose we have an integer solution x . We rewrite our equation as

$$y^2 - (x^3 + 3/2x^2 + 3/8x + 15/16)^2 = -61/64x^2 - 45/64x + 31/256,$$

and then multiply by 256 to clear denominators. Write

$$g = 16x^3 + 24x^2 + 6x + 15, \quad h = -244x^2 - 180x + 31, \quad Y = 16y.$$

Then $(Y - g)(Y + g) = h$, and (unless one of the factors is zero) $|Y - g| \leq |h|$ and $|Y + g| \leq |h|$. Note that neither factor can be zero since h has no integer roots. After combining our inequalities, we see that

$$|2g| = |(Y + g) - (Y - g)| \leq 2|h|.$$

Hence $|g| \leq |h|$. As the degree of g is larger than h , we get a small bound on x . A naive point search shows that $F_2(\mathbb{Z}) = \{\infty^\pm, (-1, \pm 1), (0, \pm 1), (-2, \pm 1)\}$.

As for the full rational points, this is (at the moment) beyond reach; for an explanation, see the remark at the end of this section. At best, we can be sure that there are no unknown rational points up to a very large height (on the order of 10^{100}) by running the Mordell-Weil sieve. Since our curve is of genus 2, we can use explicit bounds between the Weil and canonical heights to compute a basis of $J(F_2)(\mathbb{Q})$. It follows that $J(F_2)(\mathbb{Q})$ has a basis

$$P_1 = [(0, -1) + (-1, -1) - \infty^- - \infty^+], P_2 = [(0, -1) + (0, -1) - \infty^- - \infty^+], P_3 = [(0, 1) + (-2, 1) - \infty^- - \infty^+],$$

which we use while sieving with Magma; see [3] for a full discussion of the Mordell-Weil Sieve, or [22] for a basic introduction; a code for magma maybe found at

<http://homepages.warwick.ac.uk/maseap/progs/intpoint/>

3: For F_3 and subsequent curves, we will use unramified covers to determine the rational points. Note that the resultant $\text{Res}(x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1, -x) = 1$, and we study the curves

$$D^{(d)} : du^2 = x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1, \quad dv^2 = -x \quad d \in \{\pm 1\}.$$

They are $\mathbb{Z}/2\mathbb{Z}$ -covers of F_3 . Moreover, every rational point on F_3 lifts to one on some $D^{(d)}$; see Example 9 in [22]. If $d = -1$, then the curve $du^2 = x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1$ has no rational points, since it has no points in \mathbb{F}_3 . Hence $D^{(-1)}$ has no points. On the other hand, if $d = 1$, then our description of $F_2(\mathbb{Z})$ shows that $F_3(\mathbb{Z}) = \{\infty, (-1, \pm 1), (0, 0)\}$. Moreover, if we assume that there are no unknown points in $F_2(\mathbb{Q})$, then we have found all of the rational points on F_3 . In any case $0, -1 \notin S^{(4)}$, and so F_3 contribute no integers to $S^{(4)}$.

4: Similarly for F_4 we have the covers

$$D^{(d)} : du^2 = (x+1) \cdot (x^3 + 2x^2 + x + 1), \quad dv^2 = x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1, \quad d \in \{\pm 1\}.$$

Again, if $d = -1$, then one sees that there are no rational points on $D^{(d)}$ by looking in \mathbb{F}_3 . When $d = 1$, the second defining equation of D is that of F_2 , and so we use our description of $F_2(\mathbb{Z})$ to show that $F_4(\mathbb{Z}) = \{\infty^\pm, (-1, 0), (0, \pm 1), (-2, \pm 1)\}$. Under the assumption that there are no unknown rational points on F_2 , we can conclude that $F_4(\mathbb{Z}) = F_4(\mathbb{Q})$. In any event $0, -1, -2 \notin S^{(4)}$, and so F_4 also contributes nothing to $S^{(4)}$.

5: The Jacobian of F_5 has rank zero, and we easily determine that $F_5(\mathbb{Q}) = \{\infty^\pm, (0, 0)\}$. Since $0 \notin S^{(4)}$, we conclude that F_5 contributes no integers to $S^{(4)}$.

6: Since $\text{Res}(x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1, x^4 + 2x^3 + x^2 + x) = -1$, the rational points on F_6 are covered by the points on

$$D^{(d)} : du^2 = x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1, \quad dv^2 = x^4 + 2x^3 + x^2 + x, \quad d \in \{\pm 1\}.$$

If $d = 1$, then the equation $u^2 = x^4 + 2x^3 + x^2 + x$ is an elliptic curve of rank zero having rational points $\{\infty^\pm, (0, 0)\}$. It follows that $D(\mathbb{Q})$ has only the points at infinity and those corresponding to $x = 0$. If $d = -1$, then $D^{(d)}$ has no points over \mathbb{F}_3 . We conclude that $F_6(\mathbb{Q}) = \{\infty^\pm, (0, 0)\}$ unconditionally, and nothing new is added to $S^{(4)}$ in this case.

7: The rational points on the final curve F_7 are covered by the points on

$$D^{(d)} : du^2 = x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1, \quad dv^2 = -(x^3 + 2x^2 + x + 1), \quad d \in \{\pm 1\}.$$

As in previous cases, if $d = -1$, then $D^{(d)}$ has no points over \mathbb{F}_3 . On the other hand, when $d = 1$ we use our description of $F_2(\mathbb{Z})$ and $F_2(\mathbb{Q})$ to determine that $F_7(\mathbb{Z}) = \{\infty, (-2, \pm 1)\}$. Like before, if there are no unknown points on F_2 , then we will have given a complete list of points on F_7 . Note that $-2 \notin S^{(4)}$ and so, in combination with the previous cases, we have shown that $S^{(4)} \cap \mathbb{Z} = \emptyset$.

□

Remark 5.1. At the moment, proving that we have determined $F_2(\mathbb{Q})$ is far beyond reach. For one, the Galois group of $x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 1$ is S_6 , from which it follows that $\text{End}(J(F_2)) \cong \mathbb{Z}$; see [24]. In particular, F_2 does not map to any lower genus (elliptic) curves. Moreover, the rank of $J(F_2)(\mathbb{Q})$ is 3, and so the method of Chabauty and Coleman is out. Furthermore, in order to use a covering collection coming from the pullback of the multiplication by 2 map on the Jacobian, one would need to determine generators of the Mordell-Weil group $E(K)$, where E is an elliptic curve defined over a number field K of degree 17, followed by Elliptic Chabauty.

Questions and Future Work: A naive point search on the relevant curves suggests that $S^{(5)}$ and $S^{(6)}$ are probably empty. In fact, if $n = 5$, then the 15 curves, corresponding to the $2^4 - 1$ quadratic subfields of K_4 , satisfy the Chabauty condition (small rank), and so proving that $S^{(5)}$ is empty may be doable.

This begs the question as to whether all n sufficiently large satisfy $S^{(n)} = \emptyset$. Is it true for all $n \geq 5$? Of course this seems beyond reach at the moment. However, the weaker statement that $S^{(n)} \cap \mathbb{Z} = \emptyset$ may be attackable if one assumes standard conjectures on the height of integer points on hyperelliptic curves relative to the size of the defining coefficients. If true, to check whether the arboreal representation is as large as possible ($\text{Gal}(f^n) \cong \text{Aut}(T_n)$ for all n) for integer values of c , one would need only verify that it holds for $n = 3$. Moreover, by our result in [10], if $c \neq 3$, we would need only check that $\text{Gal}(f^2) \cong \text{Aut}(T_2)$.

As a slightly separate question, the author wonders whether it is true that the $\gcd(5^{2^n} + 1, 13^{2^n} + 1) = 2$ for all n . This has been verified up to $n = 30$.

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